8.3 Trig Integrals (part 2)

example

$$\int \tan^3 x \, dx$$

two ways to do subs for this example

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

just like w/ $\cos x$ and $\sin x$, we want to somehow bring in something to make substitution doable

first way: $\int \tan x \cdot \tan^2 x \, dx$

$\tan^2 x + 1 = \sec^2 x$

$$= \int \tan x \cdot (\sec^2 x - 1) \, dx$$

$$= \int \tan x \cdot \sec^2 x \, dx - \int \tan x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$\int \frac{\sin x}{\cos x} dx$

$U = \cos x$

$dU = -\sin x \, dx$

$$= \dots = \boxed{\frac{1}{2} \tan^2 x + \ln|\cos x| + C}$$

the second way:

$$\int \tan^3 x \, dx$$

$\frac{d}{dx} \tan x = \sec^2 x$ $\qquad$ $\frac{d}{dx} \sec x = \sec x \tan x$

$\tan^2 x + 1 = \sec^2 x$

$$= \int \frac{\tan^3 x}{\sec x} \cdot \sec x \, dx = \int \frac{\tan^2 x}{\sec x} \cdot \tan x \sec x \, dx$$

$$= \int \frac{\sec^2 x - 1}{\sec x} \cdot \sec x \tan x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int \frac{u^2 - 1}{u} \, du = \int \left(u - \frac{1}{u}\right) du = \frac{1}{2} u^2 - \ln|u| + C$$

$$= \boxed{\frac{1}{2} \sec^2 x - \ln|\sec x| + C}$$

they look different

$$\frac{1}{2}(\tan^2 x + 1) - \ln\left|\frac{1}{\cos x}\right| + C$$

$$= \frac{1}{2} \tan^2 x - \ln(\cos^{-1} x) + \frac{1}{2} + C$$

$$= \frac{1}{2} \tan^2 x + \ln(\cos x) + B$$ constant

example

$$\int \frac{1}{\sec x - 1}\, dx$$

another way to handle $\sec x$ and $\tan x$: turn into $\sin x$ and $\cos x$

$$= \int \frac{1}{\frac{1}{\cos x} - 1}\, dx = \int \frac{1}{\frac{1}{\cos x} - 1} \cdot \frac{\cos x}{\cos x}\, dx$$

$$= \int \frac{\cos x}{1 - \cos x}\, dx = \int \frac{\cos x}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}\, dx$$

$$= \int \frac{\cos x + \cos^2 x}{1 - \cos^2 x}\, dx = \int \frac{\cos x + \cos^2 x}{\sin^2 x}\, dx$$

$$= \int \frac{\cos x}{\sin^2 x}\, dx + \int \frac{\cos^2 x}{\sin^2 x}\, dx$$

$$= \int \frac{\cos x}{\sin^2 x}\, dx + \int \frac{1 - \sin^2 x}{\sin^2 x}\, dx = \int \frac{\cos x}{\sin^2 x}\, dx + \int \frac{1}{\sin^2 x}\, dx - \int 1\, dx$$

$$= \underbrace{\int \frac{\cos x}{\sin^2 x}\,dx}_{\substack{u=\sin x \\ du=\cos x\,dx}} + \underbrace{\int \csc^2 x\,dx}_{-\cot x} - \underbrace{\int 1\,dx}_{x}$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\vdots$$

$$= \boxed{-\frac{1}{\sin x} - \cot x - x + C}$$

strategy for $\int \tan^m x \sec^n x\, dx$

if n is even and positive $\rightarrow$ save $\sec^2 x$ and use $u = \tan x$ along with identities

sec x has pos. even power

if m is odd and positive $\rightarrow$ save $\sec x \tan x$ and use $u = \sec x$ along w/ identities

tan x has pos. odd power

example $\int \tan x \sec^7 x\, dx$

here, tan x has pos. odd power (1)

save sec x tan x

$$= \int \sec^6 x \cdot \sec x \tan x\, dx \qquad u = \sec x \quad du = \sec x \tan x\, dx$$

$$= \int u^6\, du = \frac{1}{7} u^7 + C = \boxed{\frac{1}{7}\sec^7 x + C}$$

<u>example</u> $\int \tan^5 x \sec^6 x$

tan x has pos. odd, sec x has pos. even
→ can use either strategy

try saving $\sec^2 x$, $u = \tan x$

$$\int \tan^5 x \sec^4 x \sec^2 x \, dx$$

↳ $(\sec^2 x)^2 = (\tan^2 x + 1)^2$

$$= \int \tan^5 x (\tan^2 x + 1)^2 \sec^2 x \, dx$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$= \int u^5 (u^2+1)^2 \, du = \int u^5 (u^4 + 2u^2 + 1) \, du$$

$$= \int (u^9 + 2u^7 + u^5) \, du = \frac{1}{10} u^{10} + \frac{1}{4} u^8 + \frac{1}{6} u^6 + C$$

$$= \boxed{\frac{1}{10} \tan^{10} x + \frac{1}{4} \tan^8 x + \frac{1}{6} \tan^6 x + C}$$

$$\int \tan^5 x \sec^6 x \, dx$$ now try saving $\sec x \tan x$ and $u = \sec x$

$$= \int \tan^4 x \sec^5 x \sec x \tan x \, dx$$

$\hookrightarrow (\tan^2 x)^2 = (\sec^2 x - 1)^2$

$$= \int (\sec^2 x - 1)^2 \sec^5 x \sec x \tan x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int (u^2 - 1)^2 u^5 \, du = \int (u^4 - 2u^2 + 1) u^5 \, du$$

$$= \int (u^9 - 2u^7 + u^5) \, du = \frac{1}{10} u^{10} - \frac{1}{4} u^8 + \frac{1}{6} u^6 + C$$

$$= \boxed{\frac{1}{10} \sec^{10} x - \frac{1}{4} \sec^8 x + \frac{1}{6} \sec^6 x + C}$$